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THE FORMATION OF A ONE-DIMENSIONAL RESIDUAL STRESS FIELD IN THE NEIGHBOURHOOD OF A CYLINDRICAL DEFECT IN THE CONTINUITY OF AN ELASTOPLASTIC MEDIUM†

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The one-dimensional finite deformations of the elastoplastic material of a thin-walled tube accompanying the loading of its external cylindrical surface are investigated. It is observed that, under conditions of total unloading when the external pressure is removed, the onset of a repeated plastic flow is possible, which changes the form of the distribution of the residual stresses considerably. The effect of the adaptability of an elastoplastic body with a cylindrical surface to repeated loading is pointed out. © 2003 Elsevier Science Ltd. All rights reserved.

The theory of finite elastoplastic deformation, which has been previously constructed [1–3], has been used to investigate the irreversible deformation of the material of a thin-walled tube by a pressure on its external surface. This mathematical model differs from earlier models [4–7] in that, when the overall deformations are separated into reversible (elastic) and irreversible (plastic) components, these components are determined not by algebraic relations but by differential relations, that is, transport equations [18] for the elastic and plastic deformation tensors are obtained. In particular, the result of unloading is independent of the path of the process in stress space [1, 2] in this method. This fact has been used [9] to calculate residual stresses by solving problem of the equilibrium of a body when there are no external actions but with accumulated reversible deformations.

It is noted below that the fact that the result of the unloading process is independent of its path in stress space, which is a property of the model used, still does not guarantee that the result of the unloading can be considered as the resultant equilibrium state. When the level of accumulated, irreversible deformations exceeds a certain critical value, their interdependence with the reversible deformations leads to the occurrence of a repeated plastic flow accompanying the continuing overall unloading of the body. This effect necessitates a change in the formulation of the corresponding boundary-value problem such that a subsequent unloading deformation with a developing plastic domain has to be considered as a process in time. This change in the formulation of the problem is the subject of this paper. The effect of residual stresses on the elastoplastic deformation process accompanying repeated loading is also considered.

1. INITIAL MODEL RELATIONS

The model of finite elastoplastic deformations used is described in detail in [2, 3], and we shall therefore only discuss the main relations which are subsequently required. Only total deformations can be directly measured; their separation into reversible (elastic) and irreversible (plastic) components, however reasonable the grounds for such a separation would appear here, is in essence associated with the arbitrariness of the investigator constructing the model. The following considerations serve as a basis for the separating the overall Almansi deformations d_{ij} into reversible e_{ij} and irreversible p_{ij} components in existing theories: during unloading, the rate of plastic deformations ε_{ij}^p vanishes and the change in the components of the plastic deformation tensor P_{ij} is only associated with rigid rotations $P_{ij}(t) = z_{ki}P_{km}(t_0)z_{mj}$, where z_{ik} are the components of an orthogonal tensor, and the instant if time $t = t_0$ can be associated with the instant at which the unloading process starts. It is well known [1] that such a change in the components p_{ik} is equivalent to the differential relation

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$$\frac{dp_{ij}}{dt} = r_{ik}p_{kj} - p_{ik}r_{kj}, \quad r_{ik} = -r_{ki} = -z_{mi}\frac{dz_{mk}}{dt} \tag{1.1}$$

When account is taken of this, the reversible and irreversible deformations are defined by the transport equations

$$\begin{aligned} dp_{ij}/dt &= \varepsilon^p_{ij} - \varepsilon^p_{ik}p_{kj} - p_{ik}\varepsilon^p_{kj} + r_{ik}p_{kj} - p_{ik}r_{kj} \\ de_{ij}/dt &= \varepsilon_{ij} - \varepsilon^p_{ij} - \frac{1}{2}(e_{ik}v_{k,j} + v_{k,i}e_{kj} - r_{ik}e_{kj} + e_{ik}r_{kj} - \varepsilon^p_{ik}e_{kj} - \varepsilon_{ik}\varepsilon^p_{kj}) \end{aligned} \tag{1.2}$$

In (1.2), Euler's method of specifying the motion of a medium $a_i = a_i(x_1, x_2, x_3, t)$ is adopted and the relations

$$\begin{aligned} &v_i = \frac{\partial u_i}{\partial t} + v_j u_{i,j}, \quad \varepsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \\ &u_i(x_1, x_2, x_3, t) = x_i - a_i(x_1, x_2, x_3, t) \\ &r_{ij} = w_{ij} + A^{-1}\{B^2(\varepsilon_{ik}e_{kj} - e_{ik}\varepsilon_{kj}) + B(\varepsilon_{ik}e_{km}e_{mj} - e_{ik}e_{km}\varepsilon_{mj}) + \\ &+ e_{ik}\varepsilon_{km}e_{ms}e_{sj} - e_{ik}e_{km}\varepsilon_{ms}e_{sj}\} \\ &A = 8 - 8L_1 + 3L_1^3 - L_2 - \frac{1}{3}L_1^3 + \frac{1}{3}L_3, \quad B = 2 - L_1 \\ &L_1 = e_{ii}, \quad L_2 = e_{ij}e_{ji}, \quad L_3 = e_{ij}e_{jk}e_{ki}, \quad w_{ij} = \tfrac{1}{2}(v_{i,j} - v_{j,i}) \end{aligned}$$

According to Eqs (1.2), the separation of the Almansi deformation tensor d_{ij} into its reversible component e_{ij} and irreversible component p_{ij} satisfies the relation

$$d_{ij} = e_{ij} + p_{ij} - 1/2e_{ik}e_{kj} - e_{ik}p_{kj} - p_{ik}e_{kj} + e_{ik}p_{ks}e_{sj} \tag{1.3}$$

The differential relations (1.2) are substantiated [3, 8] by the systematic use of the formalism of non-equilibrium thermodynamics.

As previously [3], we shall also adopt the simplifying hypothesis that the free energy F is independent of the irreversible deformations; the Murnaghan formula

$$\sigma_{ij} = -P\delta_{ij} + \frac{\partial W}{\partial e_{im}}(\delta_{mj} - e_{mj}) \tag{1.4}$$

will then be a consequence of the law of conservation of energy.

When there are no irreversible deformations, the Murnaghan formula takes the form

$$\sigma_{ij} = -P\delta_{ij} + \frac{\partial W}{\partial d_{im}}(\delta_{mj} - 2d_{mj}) \tag{1.5}$$

In relations (1.4) and (1.5), P is the additional hydrostatic pressure which arises as a consequence of the assumed condition that the medium is incompressible. We assume the elastoplastic medium to be isotropic and, when $W = W(L_1, L_2)$, we use the relation

$$\begin{aligned} &F(d_{ij}) = \rho_0^{-1}W(L_1, L_2) \\ &W = (a-\mu)L_1 + aL_2 + bL_1^2 - \chi L_1 L_2 - \theta L_1^3, \quad L_1 = d_{ii}, \quad L_2 = d_{ij}d_{ji} \end{aligned} \tag{1.6}$$

The quantities μ, a, b, χ, θ are assumed to be constants of the medium, and ρ_0 is the density of the medium in the undeformed state. When $b = \chi = \theta = 0$, the well-known Moony potential follows from relations (1.6) and if also $a = 0$, then we obtain Treloar's elastic potential. When there are irreversible deformations in the medium, it is necessary to replace the invariants L_1 and L_2 with the invariants of the reversible deformation tensor e_{ij}

$$I_1 = e_{ii} - \frac{1}{2}e_{ij}e_{ji}, \quad I_2 = e_{ij}e_{ji} - e_{is}e_{sj}e_{ji} + \frac{1}{4}e_{is}e_{sk}e_{kj}e_{jt}$$

This choice of invariants ensures that we can take the limit in calculations of the stresses using formulae (1.4) and (1.5) when the irreversible deformation p_{ij} tend to zero.

We will assume that the process of plastic flow is ideal and that the accumulation of irreversible deformations occurs when [10]

$$f(\sigma_{ij}) = k(df/d\sigma_{ij})\varepsilon_{ij} > 0 \tag{1.7}$$

The conditions of the Mises maximum principle

$$(\sigma_{ij} - \sigma_{ij}^*)\varepsilon_{ij}^p > 0 \tag{1.8}$$

are also adopted, where σ_{ij}^* is any stressed state which is permitted by the given loading function $f(\sigma_{ij}^*) \leqslant k$. In this case, the associated law of plastic flow

$$\varepsilon_{ij}^p = \lambda\partial f/\partial\sigma_{ij}, \quad \lambda = \lambda(\varepsilon_{ij}^p) > 0 \tag{1.9}$$

follows.

In the calculations, the condition of plasticity of Tresca's maximum shear stress was used as the loading surface $f(\sigma_{ij}) = k$.

2. FORMULATION OF THE PROBLEM. INITIAL ELASTIC EQUILIBRIUM

An infinitely long cylinder of external radius R_0, made of an elastoplastic material is considered in which there is a cylindrical cavity of radius r_0. The cylindrical surfaces $r = R_0$ and $r = r_0(r_0 \ll R_0)$ correspond to the free state of such a thin-walled tube. The external cylindrical surface $r = R$ is loaded while the internal surface $r = s$ remains load-free. An elastic equilibrium is maintained until the component of the stress tensor σ_{rr} in the cylindrical system of coordinates (r, θ, z), which is subsequently used, exceeds its threshold value P_0 at the boundary $r = R$. When

$$\sigma_{rr}(R) = -P_0 \tag{2.1}$$

the stressed state at the internal surface of the tube $r = s_0$ reaches the loading surface for the first time

$$\sigma_{rr} - \sigma_{\theta\theta} = 2k \tag{2.2}$$

that is, the condition

$$\sigma_{\theta\theta}(s_0) = -2k \tag{2.3}$$

is satisfied.

Plastic flow of the material begins from this state of elastic equilibrium and it is therefore necessary to calculate the parameters of this state for the subsequent analysis.

The condition of incompressibility of the material leads, in the case being considered, to an equation in the sole non-zero component of the displacement vector $u_r = u(r)$

$$(1 - u')(1 - r^{-1}u) = 1$$

(the prime denotes a partial derivative with respect to r).

The function

$$u = r - (r^2 + \varphi(t))^{1/2}, \quad \varphi(t) = R_0^2 - R^2(t) = r_0^2 - s^2(t) \tag{2.4}$$

is the solution of this equation.

Here, $R(t)$ and $s(t)$ are the actual values of the radii of the external and internal cylindrical surfaces respectively. Under equilibrium conditions $\varphi(t) = \text{const}$, but the time-dependence is important during the subsequent plastic flow.

For the components of the Almansi finite deformation tensor we find

$$d_{rr} = u' - \frac{1}{2}u'^2 = \frac{1}{2}(1 - \eta^{-1}), \quad d_{\theta\theta} = r^{-1}u - \frac{1}{2}r^{-2}u^2 = \frac{1}{2}(1 - \eta)$$
$$\eta = 1 + (R_0^2 - R^2)r^{-2} \tag{2.5}$$

Substituting expression (2.3) into relations (1.4) and (1.6) we can calculate the stresses in the medium apart from the unknown function $p(r)$

$$\begin{aligned}
&\sigma_{rr} = -p(r)+\xi(\eta), \quad \sigma_{\theta\theta} = -p(r)+\xi(\eta^{-1})\\
&p(r) = -P+\partial W/\partial L_1, \quad \xi(\eta) = a_1(1-\eta^{-3})+a_2(1-\eta^{-2})+\\
&+a_3(1-\eta^{-1})+a_4(\eta-1)+a_5(\eta^2-1)\\
&a_1 = \frac{3}{4}(\chi+\theta), \quad a_2 = a+b-\frac{3}{4}(3\chi+5\theta)\\
&a_3 = \mu-2a-3b+\frac{5}{2}(\chi+3\theta), \quad a_4 = b-\frac{3}{4}(\chi+5\theta), \quad a_5 = \frac{1}{4}(\chi+3\theta)
\end{aligned} \tag{2.6}$$

The function $p(r)$ is found by integrating the equilibrium equation

$$\sigma'_{rr}+(\sigma_{rr}-\sigma_{\theta\theta})r^{-1} = 0 \tag{2.7}$$

with the boundary condition $\sigma_{rr}(s_0) = 0$. We obtain

$$\begin{aligned}
&p(r) = \xi(\eta)+\zeta(x,\eta)\\
&\zeta(x,\eta) = a_6\ln(x\eta^{-1})+a_7(x-\eta)+a_8((x-1)^2-(\eta-1)^2)+\\
&+a_9((x-1)^3-(\eta-1)^3)+a_{10}(\eta^{-1}-x^{-1})+a_{11}((1-\eta^{-1})^2-(1-x^{-1})^2)\\
&a_6 = \frac{1}{2}\left(\mu-a-b+\frac{1}{2}(\chi+3\theta)\right), \quad a_7 = \frac{1}{2}\mu, \quad a_8 = \frac{1}{4}\left(a+b+\frac{1}{4}(\chi-3\theta)\right)\\
&a_9 = \frac{1}{6}a_1, \quad a_{10} = \frac{1}{2}\left(a+b-\frac{1}{2}(\chi+3\theta)\right), \quad a_{11} = \frac{1}{4}a_1, \quad x = \eta(s_0) = r_0^2s_0^{-2}
\end{aligned} \tag{2.8}$$

Relations (2.5), (2.6) and (2.8) solve the problem, apart from a single unknown quantity s_0. The boundary condition (2.1) enables one to associate this parameter with the value of the loading pressure P_0, and condition (2.3) enables us to calculate the value of P_0 or s_0 at which the condition for irreversible deformation is satisfied. This last condition takes the form of an algebraic equation in x

$$\begin{aligned}
&c_1(x-x^{-1})+c_2(x^2-x^{-2})+a_1(x^3-x^{-3}) = 2k\\
&c_1 = \mu-2a-2b+\frac{1}{4}(7\chi+15\theta), \quad c_2 = a+b-2\chi-3\theta
\end{aligned} \tag{2.9}$$

Solving Eq. (2.9), we find the value of x and, consequently, P_0 and s_0 such that the Tresca plasticity condition is satisfied on the internal cylindrical surface $r = s_0$. The solution of the subsidiary elastic problem can then be considered as having been completed.

3. PLASTIC FLOW

We will now consider yet another subsidiary problem within the framework of which it is convenient to write the majority of the relations required in the subsequent description. We will assume that, at the instant of time $t = 0$, the boundary of the cylindrical cavity occupies the position $r = s_0$, which has been calculated using Eq. (2.9). This means that the stress σ_{rr} on the external boundary of the body corresponds to the value P_0 of the external pressure. Suppose the external pressure subsequently increases

$$\sigma_{rr}|_{r=R(t)} = -P_0-g(t), \quad g(t)>0, \quad g(0) = 0 \tag{3.1}$$

Plastic deformation occurs in the domain $s(t) \leqslant r \leqslant r_1(t)$ at any successive instant of time and everywhere in this domain $\sigma_{rr}-\sigma_{\theta\theta} = 2k$. The function $r_1(t)$ specifies the motion of the boundary of the zone plastic flow. The equilibrium equation now has to be replaced by the equation of motion of the medium

$$\sigma'_{rr} + (\sigma_{rr} - \sigma_{\theta\theta})r^{-1} = -\frac{1}{2}\rho_0\left(\ddot{\varphi}r^{-1} + \frac{1}{2}\dot{\varphi}^2 r^{-3}\right) \tag{3.2}$$

(the dot denotes a derivative with respect to t).

Relations (2.2) have been used here to calculate of acceleration component. Equation (3.2) has to be integrated separately in the elastic and plastic domains taking account of boundary condition (3.1) and the condition that σ_{rr} is equal to zero when $r = s(t)$. In this manner we find that, in the domain $s(t) \leqslant r \leqslant r_1(t)$

$$\begin{aligned} \sigma_{rr} &= 2k\ln(s(t)r^{-1}) + S(\varphi, s(t), r) \\ S(\varphi, s(t), r) &= \frac{1}{2}\rho_0\left[\ddot{\varphi}\ln(s(t)r^{-1}) + \frac{1}{4}\dot{\varphi}^2(r^{-2} - s^{-2}(t))\right] \end{aligned} \tag{3.3}$$

In the domain of elastic deformation $r_1(t) \leqslant r \leqslant R(t)$, we obtain

$$\sigma_{rr} = \zeta(\eta, 1 + \varphi(t)R^{-2}(t)) - P_0 - g(t) + S(\varphi, R(t), r) \tag{3.4}$$

At the boundary of the plastic domain $r = r_1(t)$, the values of σ_{rr}, which are defined by expressions (3.3) and (3.4), are necessarily identical. Moreover, the stress σ_{rr}, calculated using formula (3.4), and $\sigma_{\theta\theta}$, which is given by a similar relation, are related by the plasticity condition (2.2). In this way, $r_1(t)$ is expressed in terms of the function $\varphi(t)$, and the ordinary differential equation

$$\begin{aligned} &\zeta(y, 1 + \varphi(t)R^{-2}(t)) - k\ln[(r_0^2 - \varphi(t))(y-1)\varphi^{-1}(t)] - \\ &- P_0 - g(t) + S\left(\varphi, (R_0^2 - \varphi(t))^{1/2}\, (r_0^2 - \varphi(t))^{1/2}\right) = 0 \\ &y = \eta|_{r = r_1(t)} = r_1^{-2}(t)\varphi(t) + 1 \end{aligned} \tag{3.5}$$

is obtained for this function

We will assume the following initial conditions

$$\varphi(0) = r_0^2 - s_0^2 = r_0^2(x-1)x^{-1}, \quad \dot{\varphi}(0) = 0 \tag{3.6}$$

for differential equation (3.5).

Equation (3.5) with initial conditions (3.6) can be solved numerically and the stresses in the medium can then be calculated using formulae (3.3) and (3.4) and similar relations for $\sigma_{\theta\theta}$, which are not written out here. The calculation of the reversible and irreversible deformations in the domain $s(t) \leqslant r \leqslant r_1(t)$, which are required in the subsequent treatment, turn out to be an independent problem in this case. When calculating them, in has to be taken into account that the reversible deformations at each point change up to the time when the boundary of the plastic domain reaches this point and, subsequently, on account of the ideal nature of the plastic flow, they remain unchanged until the end of the loading. The boundary of the plastic domain changes during the deformation process so that $r = s_0$ when $p = P_0$ and $r = r_1$ when $p = P_1$ (the final state). If r is the coordinate of a point at the final instant of deformation, r_{t0} is its coordinate in the free state and r_t is the coordinate of the same point at the instant when plastic flow begins in its neighbourhood, then

$$\begin{aligned} r &= (\gamma_t + r_t^2 - \gamma_1^2)^{1/2}, \quad r_{t0}^2 = \gamma_t + r_t^2 \\ \gamma_t &= r_{t0}^2 - r_t^2 = r_0^2 - s_t^2 = R_0^2 - R_t^2 \\ \gamma_1 &= r_{t0}^2 - r^2 = r_0^2 - s_1^2 = R_0^2 - R_1^2 \end{aligned} \tag{3.7}$$

Here γ_1 and γ_t are the values of the function $\varphi(t)$ at the instant the deformation process ceases and at the current instant of time (when $r = r_1$) respectively. The functions γ_1 and γ_t are determined from the solution of Eq. (3.5) with the conditions (3.6). Relations (3.7) associate the initial (material or Lagrangian) coordinate of a point r_{t0} with its spatial (Eulerian) coordinate r in terms of the spatial coordinate of the same particle r_t at the instant of the plastic boundary approaches it. If we put $g(t) = \lambda t$ in condition (3.1), we obtain

$$\gamma_1 = \varphi(P_1 - P_0)\lambda^{-1}, \quad \gamma_t = \varphi(P_t - P_0)\lambda^{-1}; \quad P_0 < P_t \le P_1$$

Using the condition for the plastic flow process to be ideal, we find the reversible deformations when $s_1(t) \leqslant r \leqslant r_1(t)$

$$e_{rr} = 1 - x^{-1/2}, \quad e_{\theta\theta} = 1 - x^{1/2}; \quad x = 1 + \gamma_t(r^2 + \gamma_1 - \gamma_t)^{-1} \tag{3.8}$$

The irreversible deformations are calculated from (1.3) using the known reversible and total deformations

$$p_{rr} = \frac{1}{2}x(\gamma_1(r^2 + \gamma_1)^{-1} + x^{-1} - 1), \quad p_{\theta\theta} = \frac{1}{2}x^{-1}(x - 1 - \gamma_1 r^{-2}) \tag{3.9}$$

If the final value of the specified external pressure P_1 does not change later, the body with the accumulated plastic deformations (3.9) will come into equilibrium.

4. LOADING OF THE MEDIUM. RESIDUAL STRESSES

We unload the body by changing the external pressure to a certain value $P_* < P_1$. A threshold value for the loading pressure $P_1 = P_1^*$ exists such that, if $P_1 \leqslant P_1^*$, the value of P_* can be taken as being equal to zero and the final unloaded state can be considered as the equilibrium state when the values of σ_{rr} and $r = R_p$ are zero (the subscript p corresponds to the unloaded state). This is associated with the fact that, in the model of finite elastoplastic deformations which is being used, this state is independent of the path of the unloading process in stress space.

When $P_1 > P_1^*$, the condition for the final state to be independent of the unloading process cannot be made use of on account of the fact that only $\sigma_{rr}(R(t))$ remains equal P^* and the stressed state is defined by the equality $\sigma_{\theta\theta}(s_p) = 2k$. Note that $\sigma_{\theta\theta}$ is now a stretching stress. This state turns out to be the initial state for the subsequent plastic flow process associated with a further reduction in the external load. The overall deformations in the medium in this state are calculated using the known displacement field (2.2), where the function $\varphi(t)$ has to be replaced by its value $\gamma_p = r_0^2 - s_p^2$. If the material boundary of the plastic domain does not change during unloading, the spatial coordinate of the boundary when $p = P_2$ is calculated using the formula $r_{1p}^2 = r_1^2 + \gamma_p - \gamma_1$. In the domain $r_{1p}(t) \leqslant r \leqslant R_p$, where there are no irreversible deformations, the stresses are calculated using the known overall deformations from relations (1.5) and (1.6). The unknown function of the additional hydrostatic pressure, which appears in (1.5), is found by integrating the equilibrium equation subject to the condition $\sigma_{rr}(R_p) = -P_2$. As a result, we find

$$\begin{aligned} \sigma_{rr} &= -P_2 + \zeta(\eta, \eta_1) \\ \sigma_{\theta\theta} &= \sigma_{rr} - a_1(\eta^3 - \eta^{-3}) - c_2(\eta^2 - \eta^{-2}) - c_1(\eta - \eta^{-1}) \\ \eta &= 1 + \gamma_p r^{-2}, \quad \eta_1 = 1 + \gamma_p R_0^{-2} \end{aligned} \tag{4.1}$$

In the domain with the accumulated irreversible deformations, the stresses are calculated using the elastic deformations which have been found. The latter are calculated in terms of the known total plastic deformations, which do not change during the unloading process. The unknown function P in expression (1.4) is found by integrating the equilibrium equation with the condition $\sigma_{rr}(s_p) = 0$. Hence, the final relations

$$\begin{aligned} \sigma_{rr} &= \tau(y_1, y_2) \\ \sigma_{\theta\theta} &= \sigma_{rr} - a_1(y_2^3 - y_2^{-3}) - c_2(y_2^2 - y_2^{-2}) - c_1(y_2 - y_2^{-1}) \\ \tau(y_1, y_2) &= c_3 \ln((y_2 - 1)(y_1 - 1)^{-1}) + c_4(y_1 - y_2) + \\ &+ c_5 \ln((y_2 - 1)(y_1 - 1)^{-1} y_1 y_2^{-1}) + c_6(y_2^{-1} - y_1^{-1}) + \\ &+ c_7((y_1 - 1)^2 - (y_2 - 1)^2) + c_8((y_1 - 1)^3 - (y_2 - 1)^3) + c_9((1 - y_2^{-1})^2 - (1 - y_1^{-1})^2) \end{aligned} \tag{4.2}$$

$$y_1 = 1 + (\gamma_p - \gamma_1)s_p^{-2}, \quad y_2 = 1 + (\gamma_p - \gamma_1)r^{-2}$$

$$c_3 = \frac{1}{2}(c_1x + c_2x^2 + a_1x^3), \quad c_4 = -\frac{1}{2}(c_1x + 2c_2x^2 + 3a_1x^3)$$

$$c_5 = -\frac{1}{2}(c_1x^{-1} + c_2x^{-2} + a_1x^{-3}), \quad c_6 = -\frac{1}{2}(c_2x^{-2} + a_1x^{-3})$$

$$c_7 = -\frac{1}{4}(c_2x^2 + 3a_1x^3), \quad c_8 = -\frac{1}{6}a_1x^3, \quad c_9 = \frac{1}{4}a_1c^{-3}$$

can be obtained for the stresses in the domain $s_p(t) \leqslant r \leqslant r_{1p}$.

According to the condition which has been adopted, the stressed state of the medium, specified by relations (4.2), must satisfy the condition $\sigma_{\theta\theta}(s_p) = 2k$. The algebraic equation

$$a_1(z^3 - z^{-3}) + c_2(z^2 - z^{-2}) + c_1(z - z^{-1}) = -2k, \quad z = xy_1^{1/2} \tag{4.3}$$

follows from this.

The solution of this equation gives the magnitude of z and, consequently, the value of s_p which corresponds to the onset of repeated plastic flow. This value is uniquely related to the value of P_2 for the loading pressure; the latter is calculated from the condition for the stresses (4.1) and (4.2) to be equal on the boundary of the plastic domain $r = r_{1p}$.

It has already been noted that the calculated equilibrium state is the initial state for the subsequent process of the irreversibility of deformation in which

$$\begin{aligned} &\sigma_{rr}|_{r=R(t)} = -P_2 + h(t), \quad h(0) = 0, \quad h(t) > 0 \\ &(\sigma_{rr} - \sigma_{\theta\theta})|_{s(t) \le r \le r_2(t)} = -2k, \quad \sigma_{rr}|_{r=s(t)} = 0 \end{aligned} \tag{4.4}$$

The equilibrium equation now has to be replaced by the equation of motion (3.2) with a new arbitrary function $\psi(t)$. The motion of the boundary of the new plastic domain $r = r_2(t)$ is given by the relation

$$r_2 = (x(\psi(t) - \gamma_1))^{1/2}(z - x)^{-1/2} \tag{4.5}$$

By integrating the equations of motion in each of the three domains

$$s(t) \le r \le r_2(t), \quad r_2(t) \le r \le r_1(t), \quad r_1(t) \le r \le R(t)$$

subject to the conditions that the elastic deformations do not change in the first of these domains and that the plastic deformations do not change in the second, the stresses in each of the deformation domains are determined using boundary conditions (4.4). Here, we shall only write out the relation $\sigma_{rr} = \sigma_{rr}(r, t)$; the relations for $\sigma_{\theta\theta}(r, t)$ are completely analogous to it.

In the domain $s(t) \leqslant r \leqslant r_i(t)$, we have

$$\sigma_{rr} = 2k\ln(rs^{-1}(t)) + S(\psi, s(t), r) \tag{4.6}$$

In the domain $r_1(t) \leqslant r \leqslant R(t)$, where there are no irreversible deformations, the stresses σ_{rr} are calculated using the first expression of (4.1) in which it is necessary to add the term

$$S(\psi, (R_0^2 - \psi)^{1/2}, r) - h(t)$$

When integrating the equation of motion in the domain $r_2(t) \leqslant r \leqslant r_1(t)$, where the irreversible deformations do not change, it is necessary to use the condition for the stresses to be equal when $r = r_1(t)$ (the change in r_1 with time is solely associated with the motion of the deforming medium). As a result, we find that, in the given domain

$$\begin{aligned} &\sigma_{rr} = \tau(y_3, y_2) + h(t) - P_2 + \zeta(z_1, \eta_1) + S(\psi, (R_0^2 - \psi)^{1/2}, r) \\ &y_3 = 1 + (\psi(t) - \gamma_1)r_1^{-2}(t), \quad z_1 = 1 + \psi(t)r_1^{-2}(t) \end{aligned} \tag{4.7}$$

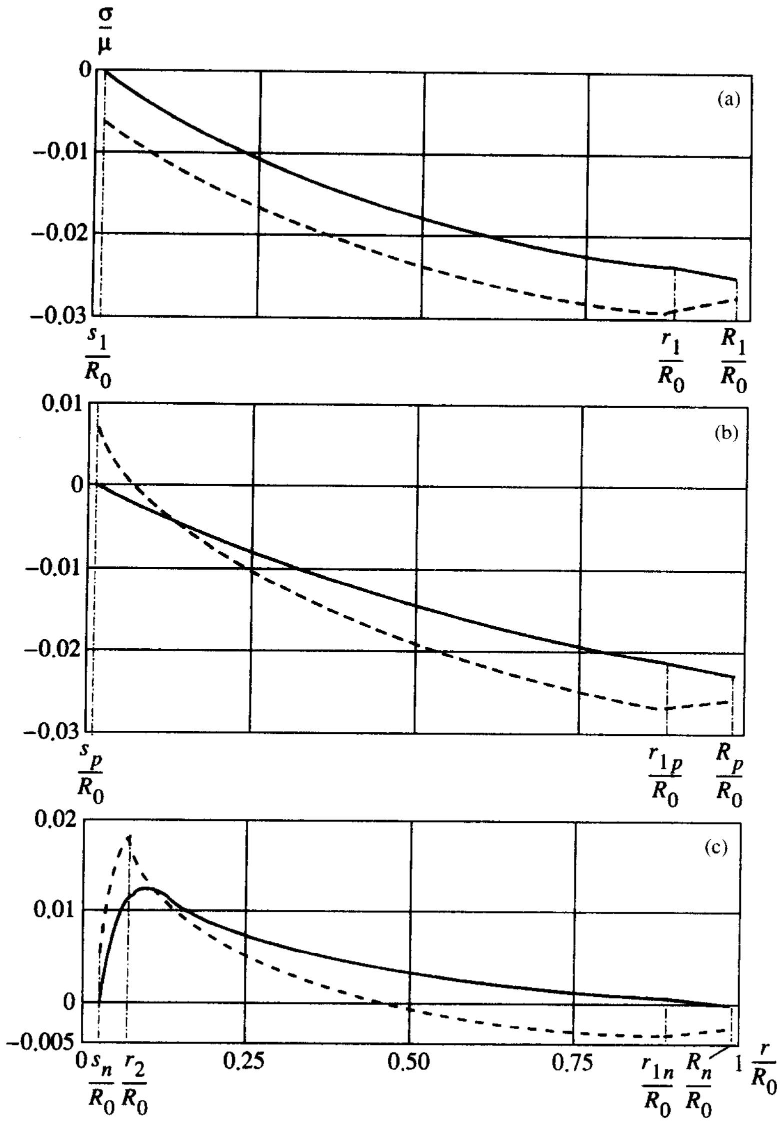

Fig. 1

The condition for the stresses to be equal on the boundary of the domain of repeated plastic flow $r = r_2(t)$ leads to the following differential equation for the function $\psi(t)$

$$\tau(y_3, zx^{-1}) + \zeta(z_1, \eta_1) - k\ln(r_2^2(r_0^2 - \psi(t))^{-1}) + h(t) - P_2 + \\ + S(\psi, (R_0^2 - \psi)^{1/2}(r_0^2 - \psi)^{1/2}) = 0 \tag{4.8}$$

The initial conditions for these equations are

$$\psi(0) = r_0^2 - s_p^2, \quad \dot{\psi}(0) = 0 \tag{4.9}$$

Hence, the solution of the problem of the unloading of the medium has been reduced to the successive solution of differential equations (3.5) and (4.8) with conditions (3.6) and (4.9) respectively. The stresses and deformations in the body are determined using the functions $\varphi(t)$ and $\psi(t)$ which have been found using this technique in accordance with the above relations. The same functions also determine the laws of motion of the boundaries of the plastic domains $r_1(t)$ and $r_2(t)$. It should be noted that the irreversible deformations in the domain $r_2(t) \leqslant r \leqslant r_1(t)$, which occur parametrically in these equations, can only be given in the form of discrete numerical blocks for subsequent calculations. Their invariance

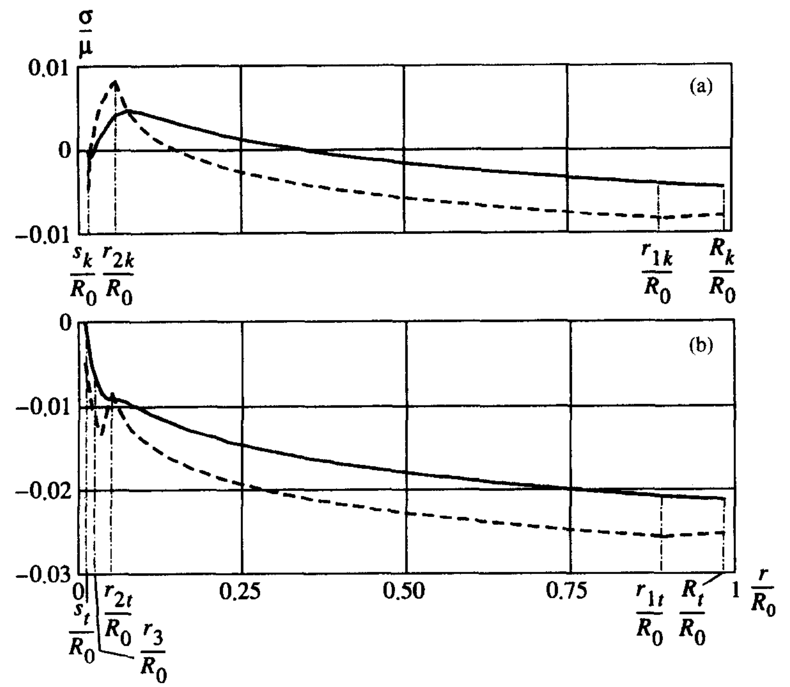

Fig. 2

at each point is guaranteed but they must be recalculated on account of the change in the spatial coordinate of the same point during the deformation process.

We will now present some typical results of calculations obtained using the following values of the constants

$$\frac{a}{\mu} = 0.9, \quad \frac{b}{\mu} = 4, \quad \frac{\chi}{\mu} = 20, \quad \frac{\theta}{\mu} = 80, \quad \frac{k}{\mu} = 0.003, \quad \frac{\lambda}{\mu} = 0.0004$$

where the shear modulus $\mu = 25 \times 10^9$ Pa. The stresses σ/μ are shown in all the figures (the stresses σ_{rr}/μ are shown by the solid curves and the stresses $\sigma_{\theta\theta}/\mu$ are shown by the dashed curves) as a function of the value of r/R_0 at different instants of deformation. The stress distribution at the final instant of active deformation when $\sigma_{rr}(R) = -P_1$ is shown in Fig. 1(a). Everywhere in the domain $s_1 < r < r_1$, the difference between the stresses $\sigma_{rr} - \sigma_{\theta\theta}$ is the same and is equal to $2k$. The stress distribution at the instant of the onset of repeated plastic flow is illustrated in Fig. 1(b). The stress $\sigma_{\theta\theta}$ in this state became a stretching stress and, on the boundary $r = s_p$, it is equal to $2k$. The final residual stresses correspond to the relations represented graphically in Fig. 1(c) $\sigma_{rr} - \sigma_{\theta\theta} = 2k$ everywhere in the domain $s_n \leqslant r \leqslant r_2$. The functions $g(t)$ and $h(t)$ were assumed to be linear in the calculations.

5. REPEATED LOADING

The initial state for repeated loading is the completely unloaded state when $\sigma_{rr}(s_n) = 0$ and $\sigma_{rr}(R_n) = 0$. The residual stress distribution in this case is shown in Fig. 1(c). Reversible deformation occurs when the external load is successively increased up to a certain value $P_3 > P_0$ and, when $\sigma_{rr}(R) = -P_3$, the stressed state on the surface $r = s_k$ (the inner surface of the tube) again reaches the yield surface: $\sigma_{\theta\theta}(s_k) = -2k$. This state is the initial state for the subsequent process of irreversible deformation and is calculated in the same way as in the preceding case, that is, the equilibrium equation is integrated in the three domains

$$s_k \le r \le r_{2k}, \quad r_{2k} \le r \le r_{1k}, \quad r_{1k} \le r \le R_k$$

where the irreversible deformations, which are unchanged during the deformation process, are calculated differently. A typical stress distribution in this state of the body is shown in Fig. 2(a). When $p > P_3$, there is an accumulation of irreversible deformations and it is therefore again necessary to treat this process in time. This leads, as earlier, to a system of equations consisting of several algebraic equations and a single differential equation. All of them are obtained by methods which are exactly the same as those demonstrated earlier, so we shall therefore only discuss the qualitative results of the calculations.

It has already been mentioned that the process of plastic flow commences when $\sigma_{rr}(R) = -P_3$, where $P_0 < P_3 < P_1$. When there is a further increase in the external pressure $\sigma_{rr}(R) = -P_3(1 + \lambda t)$ with time, the zone of plastic flow occupies the domain $s_k(t) < r < r_3(t)$ where $r_3(t)$ is the boundary of the plastic domain which is moving through the medium.

The stress distribution in the material at a certain instant of time is illustrated in Fig. 2(b). The following fact is of interest: when the boundary $r_3(t)$ reaches the surface r_2, which corresponds to the final position of the boundary of the zone of plastic flow during the unloading of the medium, the loading pressure $p(t)$ is found to be exactly equal to P_1. The domain $r_3 = r_2 < \mathrm{r} < r_1$ at this instant of time instantaneously arrives at the plastic state ($\sigma_{rr} - \sigma_{\theta\theta} = 2k$). Hence, the cycle is completed, that is, the medium arrives at the state which has been illustrated in Fig. 1(a).

If the external pressure is not increased any further and the body is again unloaded, then, as a result, we obtain the stressed state shown in Fig. 2(c). This is also typical of the changes in the radii of the boundary surfaces. We call this effect the adaptability of the medium to cyclic loading (we recall that all the calculations were carried out within the framework of the model of ideal plasticity). In order to increase the level of irreversible deformations or, what is the same thing, to reduce the radius of the cylindrical cavern in the body, it is necessary to increase the external pressure compared with the value P_1 initially reached.

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